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## Working with Complex Numbers in Mathcad

by Mathcad Staff

A complex number is an expression of the form

$$x + y \cdot i$$

where  $x$  and  $y$  are real numbers, and

$$i = \sqrt{-1}$$

For example, the following is a complex number:

$$2 + 13i$$

To enter this number, type  $2 + 13i$  in a math region.

**Note:** If the coefficient of  $i$  in a complex number is 1, you must type  $1i$  when entering the number. Otherwise, Mathcad interprets  $i$  as an undefined variable. For example, to enter  $i$  itself, type  $1i$ .

$i$

Mathcad does not display the coefficient 1. Note that  $i$  is the square root of  $-1$ .

$$i^2 = -1$$

Addition and multiplication of complex numbers is defined as follows:

$$(x + y \cdot i) + (u + v \cdot i) = (x + u) + (y + v) \cdot i$$

$$(x + y \cdot i)(u + v \cdot i) = (x \cdot u - y \cdot v) + (x \cdot v + y \cdot u) \cdot i$$

Here are some examples:

$$(2 + 3i) + (7 - 4i) = 9 - i$$

$$(2 + 3i) \cdot (7 - 4i) = 26 + 13i$$

### Real and Imaginary Parts

For a complex number  $x + yi$ ,  $x$  is called the *real* part and  $y$  is called the *imaginary* part. The functions **Re** and **Im** return the real and imaginary parts, respectively.

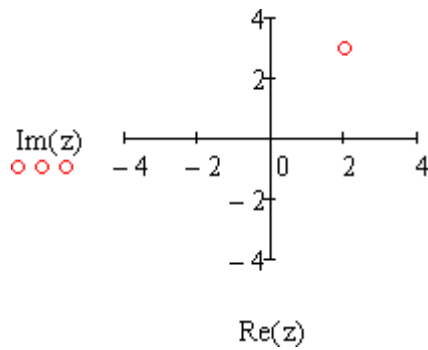
$$\text{Re}(7 - 4i) = 7$$

$$\text{Im}(7 - 4i) = -4$$

### Representing Complex Numbers as Points in the Plane

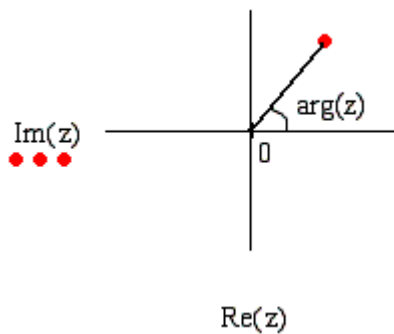
You can represent a complex number  $x + yi$  as the point  $(x, y)$  in the plane. For example,  $2 + 3i$  corresponds to the point  $(2, 3)$ , as shown in the graph below.

$$z := 2 + 3i$$



This representation is useful for visualizing operations such as multiplication on the complex numbers.

The *argument* of a complex number  $z$  is the angle in radians that the vector corresponding to  $z$  makes with the x-axis. The argument, denoted  $\arg(z)$ , is assumed to be between  $-\pi$  and  $\pi$ .



The function **arg** returns the argument of  $z$ .

$$\arg(2 + 3i) = 0.983$$

When you multiply two complex numbers, the argument of the product is the sum of their arguments, provided the sum is between  $-\pi$  and  $\pi$ . For example,

$$z := 2 + 3i$$

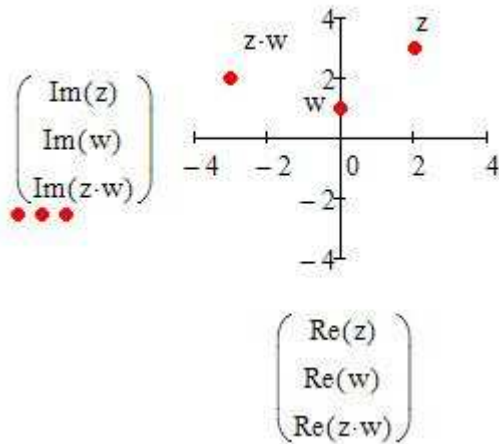
$$w := i$$

$$\arg(z) = 0.983$$

$$\arg(w) = 1.571$$

$$\arg(z) + \arg(w) = 2.554$$

$$\arg(z \cdot w) = 2.554$$



Since the argument of  $i$  is  $\pi/2$  or 90 degrees, multiplying a complex number by  $i$  rotates the corresponding point by 90 degrees in the counter-clockwise direction. The *magnitude* of a complex number is the distance from the point to the origin. You can compute the magnitude using the absolute value operator on the Calculator toolbar.

$$|2 + 3i| \rightarrow \sqrt{13}$$

The magnitude of the product of two complex numbers is the product of their magnitudes.

### The Complex Exponential Function

The usual exponential function on the real numbers can be extended to the complex numbers by the following definition.

$$e^{x+yi} = e^x \cdot (\cos(y) + \sin(y) \cdot i)$$

When the exponent is purely imaginary - that is, when its real part is 0, this equation becomes

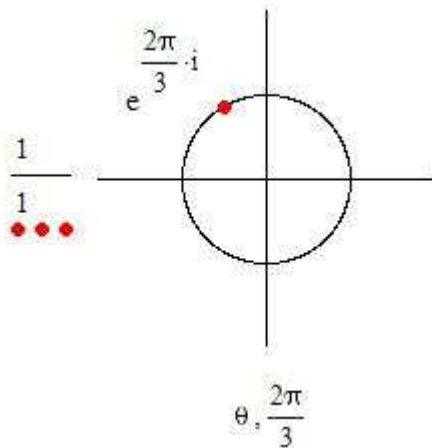
$$e^{yi} = \cos(y) + \sin(y) \cdot i$$

The magnitude of  $e^{yi}$  is

$$|\cos(y) + \sin(y) \cdot i| \rightarrow |\cos(y) + \sin(y) \cdot i| \text{ simplify } \rightarrow e^{-\text{Im}(y)}$$

So  $e^{yi}$  corresponds to a point on the unit circle in the complex plane. For example,

$$e^{\frac{\pi}{3}i} \rightarrow \frac{1}{2} + \frac{\sqrt{3}i}{2}$$



If you square a complex number  $z$  on the unit circle, the magnitude is 1, since

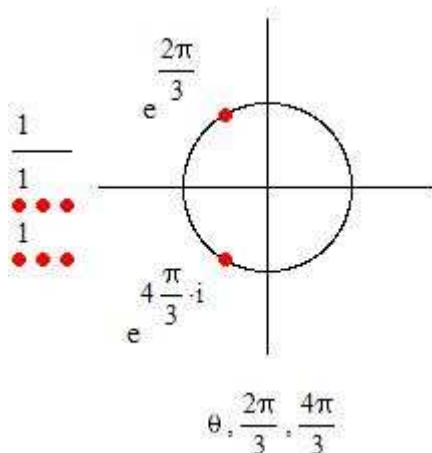
$$|z^2| = |z| \cdot |z| = 1 \cdot 1 = 1$$

The argument of  $z^2$  is twice the argument of  $z$ , since

$$\arg(z^2) = \arg(z) + \arg(z) = 2 \cdot \arg(z)$$

For example,

$$\left( e^{\frac{2\pi}{3} \cdot i} \right)^2 \text{ expand } \rightarrow -\frac{1}{2} - \frac{\sqrt{3} \cdot i}{2}$$



### Roots of Unity

In the complex numbers, the equation

$$z^n = 1$$

has  $n$  distinct solutions, called the  *$n$ th roots of unity*. These are defined as follows. Let

$$\alpha = e^{\frac{2\pi}{n} \cdot i} = \cos\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) \cdot i$$

$\alpha$  corresponds to the point on the unit circle in the

complex plane whose angle is  $2\pi/n$ .

Note that  $\alpha$  is a solution to the equation

$$x^n = 1$$

since

$$\alpha^n = \left( e^{\frac{2\pi}{n} \cdot i} \right)^n = e^{2\pi \cdot i} = \cos(2\pi) + \sin(2\pi) \cdot i = 1$$

The  $n$ th roots of unity are the numbers

$$\alpha, \alpha^2, \alpha^3, \dots, \alpha^n = 1$$

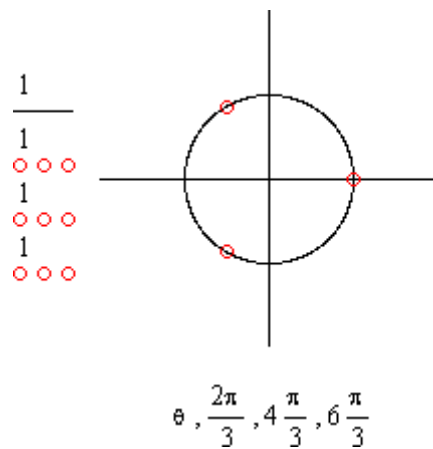
For example, if  $n = 3$ , the cube roots of unity are the numbers

$$\alpha := e^{\frac{2\pi}{3} \cdot i} \rightarrow -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

$$\alpha^2 \text{ expand} \rightarrow -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

$$\alpha^3 \text{ expand} \rightarrow 1$$

Since raising  $\alpha$  to any power  $k$  multiplies the argument of  $\alpha$  by  $k$ , the powers of  $\alpha$  are evenly spaced around the unit circle. The following graph shows the case  $n = 3$ .



$\alpha$

$$\alpha^3 = 1$$

$\alpha^2$

Note that each of these numbers is a solution to the equation

$$x^3 = 1$$

### Complex Results and the Principal Value

Using the exponentiation operator to find the  $n$ th root of a negative number often returns a complex solution, even though a real solution may be present. Consider the cube root of  $-27$ :

$$(-27)^{\frac{1}{3}} = 1.5 + 2.598i$$

even though

$$(-3)^3 = -27$$

### Principal Value

When there are multiple roots, as is the case here, the exponentiation operator returns the **principal value**, that is the value with the smallest positive angle with respect to the positive real axis in the complex plane.

Note that the **nth root** operator always defaults to the positive root, if one is available. (Note the nth root operator is on the Calculator toolbar or type [Ctrl] [\].

$$\sqrt[3]{-27} = -3$$

If there is no real root, the nth root operator defaults to the principal value.

### arg Function

The principal value is not always the positive root. Mathcad's **arg** function can be used to validate this assertion.

$\arg(z)$

returns the principal argument of  $z$  between  $-\pi$  and  $\pi$ , including  $\pi$ .

$$z := (-27)^{\frac{1}{3}}$$

$$z = 1.5 + 2.598i$$

$$\arg(z) = 0.333 \pi$$

whereas

$$\arg(-3) = \pi$$

To find all three solutions, including the positive root, you can solve for the roots symbolically. First, set up an equation to solve using any variable and the Boolean equality from the

**Boolean** toolbar or by typing [Ctrl] =

$$x^3 = 27$$

Next, use the solve keyword from the **Symbolic** toolbar to find the roots.

$$x^3 = -27 \text{ solve, } x \rightarrow \begin{pmatrix} -3 \\ \frac{3}{2} + \frac{3i\sqrt{3}}{2} \\ \frac{3}{2} - \frac{3i\sqrt{3}}{2} \end{pmatrix}$$

Note that Mathcad displays all values simultaneously, both real and complex.

It is also possible to find these roots numerically, using the **polyroots** function.

$$v := x^3 - 27 \text{ coeffs, } x \rightarrow \begin{pmatrix} -27 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{polyroots}(v) = \begin{pmatrix} -1.5 + 2.598i \\ -1.5 - 2.598i \\ 3 \end{pmatrix}$$

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